

NPTEL ONLINE CERTIFICATION COURSES

Management of Fixed Income Securities

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Lecture 21: Determination of Theoretical Spot Rate and Forward Yield Curves

CONCEPTS COVERED

- **Theoretical Spot Rate Curve**
- **Forward Yield Curve**

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KEYWORDS

- **Benchmark Yield Curve**
- **Bootstrapping**
- **Implied forward rate**

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Benchmark Yield Curve and Forward Yield Curve

- A spot yield curve in which the spot rates are estimated using bootstrapping, in turn, is referred to as a ***theoretical spot rate curve***. The theoretical spot rate curve is often used by practitioners to price financial instruments and by dealers to identify arbitrage opportunities. It represents a good estimate of the benchmark yield curve.
- Once a theoretical spot rate curve is derived, it can then be used as the base yield curve, with other yield curves generated by adding estimated risk and liquidity premiums.
- The spot rate curve can also be used to estimate implied forward rates and to generate forward yield curves.

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Benchmark Yield Curve and Forward Yield Curve

Security	Type	Maturity	Semi-Annual Coupon	Annualized YTM	Face Value	Current Price	Spot Rate
1	Zero Coupon	0.5		5%	100	97.561	5.00%
2	Zero Coupon	1.0		5.25%	100	94.9497	5.25%
3	Coupon Bond	1.5	2.75	5.5%	100	100	5.551%
4	Coupon Bond	2.0	2.875	5.75%	100	100	5.577%
5	Coupon Bond	2.5	3	6.00%	100	100	6.03%
6	Coupon Bond	3.0	3.125	6.25%	100	100	6.30%

There are two zero coupon with maturities of six months (0.5 years) and one year, trading at yields of 5% and 5.25%. These rates can be used as spot rates (S_t) for maturities of 0.5 years ($S_{0.5}$) and one year (S_1). The other bonds shown in the exhibit are coupon bonds assumed to be trading at par and therefore with yields equal to their coupon rates.

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Estimating Theoretical Yield Curve

- Find the spot rate for 1.5 years:
- Taking the coupon bond with a maturity of 1.5 years and annual coupon rate of 5.5% (semi-annual coupons of 2.75)
- Setting the par value of the 1.5-year bond equal to the present value of its cash flows discounted at known spot rates of $S_{0.5}$ and S_1 and an unknown spot rate for 1.5 years, $S_{1.5}$
- Solving for the spot rate for 1.5 years.

$$P_{1.5} = \frac{CF_{0.5}}{(1+(S_{0.5}/2))^1} + \frac{CF_{1.0}}{(1+(S_1/2))^2} + \frac{CF_{1.5}}{(1+(S_{1.5}/2))^3}$$

$$100 = \frac{2.75}{(1+(0.05/2))^1} + \frac{2.75}{(1+(0.0525/2))^2} + \frac{102.75}{(1+(S_{1.5}/2))^3}$$

$$94.705956 = \frac{102.75}{(1+(S_{1.5}/2)^3}$$

$$S_{1.5} = 2\left[\left[\frac{102.75}{94.705956}\right]^{1/3} - 1\right] = 0.0551$$

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Estimating Theoretical Yield Curve

- To obtain the spot rate for a two-year bond (S_2), we repeat the process using the two-year bond paying semi-annual coupons of 2.875 and selling at par.

- This yields a spot rate of $S_2 = 5.577\%$.

- Continuing the process with the other securities in the table, we obtain spot rates for bonds with maturities of 2.5 years and 3 years: $S_{2.5} = 6.03\%$ and $S_3 = 6.30\%$ (see the last column of the Table given in the previous slide).

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Estimating Theoretical Yield Curve

- To obtain the spot rate for a two-year bond (S_2), we repeat the process using the two-year bond paying semi-annual coupons of 2.875 and selling at par.

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Forward Yield Curves

- In general, implied semiannual forward rates on M-year bonds purchased t years from the present can be found using the following formula:

$$f_{Mt} = \left[\frac{(1+S_{M+t})^{N}}{(1+S_{t})^{i}} \right]^{1/(N-i)} - 1$$

where:

S = Semiannual spot rate
M = time to maturity in years
t = time period from the present to the forward date in years
N = number of semiannual periods to M + t
i = number of semiannual periods to t

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Forward Yield Curve

Security	Type	Maturity	Spot Rate	Semi annual
1	Zero Coupon	0.5	0.05	0.025
2	Zero Coupon	1.0	0.0525	0.02625
3	Coupon Bond	1.5	0.0551	0.02755
4	Coupon Bond	2.0	0.05577	0.027885
5	Coupon Bond	2.5	0.0603	0.03015
6	Coupon Bond	3.0	0.0630	0.0315

Using the semiannual spot rates, the implied forward rate on the 6-month bond, six months from now is $f_{Mt} = f_{0.5,0.5}$ This rate is obtained by defining the one-year spot rate as the geometric average of the current 6-month spot rate and the implied forward rate on a 6-month rate, six months forward, and then solving the equation for the implied forward rate

$$S_1 = [(1+S_{0.5})(1+f_{0.5,0.5})]^{1/2} - 1$$

$$f_{0.5,0.5} = \frac{(1+S_1)^2}{(1+S_{0.5})} - 1$$

$$f_{0.5,0.5} = \frac{(1.02625)^2}{(1.025)} - 1 = 0.027502$$

$$Annualized \quad f_{0.5,0.5} = (2)(0.027502) = 0.055003$$

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Forward Yield Curves

Similarly, the implied forward rate on a 1-year bond, purchased 6 months from now ($f_{Mt} = f_{1,.5}$) is obtained by solving the 1.5-year geometric mean for $f_{1,.5}$. M = 1 year, t = 0.5 year, N = 3 (= number of semiannual periods to M + t) i = 1 (= number of semiannual periods to t)

$$f_{Mt} = \left[\frac{(1+S_{M+t})^N}{(1+S_t)^i}\right]^{1/(N-i)} - 1$$

$$f_{1,0.5} = \left[\frac{(1+S_{1.5})^3}{(1+S_{0.5})^1}\right]^{1/(3-1)} - 1$$

$$f_{1,0.5} = \left[\frac{(1.027550)^3}{(1.025)^1}\right]^{1/2} - 1$$

$$f_{1,0.5} = 0.028827$$

$$Annualized \quad f_{1,0.5} = (2)(0.028827) = 0.057655$$

Forward Yield Curves

The implied forward rate on a 0.5-year bond, purchased one year from now is

M = 0.5 years, t = 1 year, N = 3 (= number of semiannual periods to M + t)

i = 2 (= number of semiannual periods to t)

$$f_{Mt} = \left[\frac{(1+S_{M+t})^N}{(1+S_t)^i} \right]^{1/(N-i)} - 1$$

$$f_{0.5,1} = \left[\frac{(1+S_{1.5})^3}{(1+S_1)^2} \right]^{1/(3-2)} - 1$$

$$f_{0.5,1} = \left[\frac{(1.027550)^3}{(1.02625)^2} \right]^1 - 1$$

$$f_{0.5,1} = 0.030155$$

$$Annualized \ \ f_{0.5,1} = (2)(0.030155) = 0.060310$$

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Forward Yield Curves

The implied forward rate on one year bond, purchased one year from now is :

M = 1 year, t = 1 year, N = 4 (= number of semiannual periods to M + t), i = 2 (= number of semiannual periods to t)

$$f_{Mt} = \left[\frac{(1+S_{M+t})^{N}}{(1+S_{t})^{i}}\right]^{1/(N-i)} - 1$$

$$f_{1,1} = \left[\frac{(1+S_{2})^{4}}{(1+S_{1})^{2}}\right]^{1/(4-2)} - 1$$

$$f_{1,1} = \left[\frac{(1.027885)^{4}}{(1.02625)^{2}}\right]^{1/2} - 1$$

$$f_{1,1} = 0.029523$$

$$Annualized \;\; f_{1,1} = (2)(0.029523) = 0.059045$$

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Forward Yield Curves

- If the implied forward rates six months from now are realized, then an investor with a six-month horizon, would earn the same semiannual rate of 2.5% obtained on the six-month bill by buying any maturity bond and selling it six months later at its implied forward rate.

- Similarly, if the implied forward rates one year from now are realized, then an investor with a one-year horizon would earn the same semiannual rate of 2.6255% obtained on the one-year by buying any maturity bond and selling it one year later at its implied forward rate.

- One of practical uses of the forward curve is that it can be used to identify the hedgable rates useful in determining alternative investments.

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CONCLUSIONS

- A spot yield curve in which the spot rates are estimated using bootstrapping is referred to as a theoretical spot rate curve and it is used as a benchmark yield curve
- The spot rate curve is used to estimate implied forward rates and to generate forward yield curves.
- The forward curve can be used to identify the hedgable rates, which is useful in determining alternative investments.

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REFERENCES

- Johnson, R.S. (2010), Bond Valuation, Selection, and Management, Second Edition, John Wiley & Sons, Inc., Hoboken, New Jersy.
- Fabozzi, J. Frank and Mann, V. Steven (2005): The Hand Book of Fixed Income Securities, Tata McGraw-Hill, 7th Edition.

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Lecture 22: Interest Rate Risk

CONCEPTS COVERED

- **Market Risk**
- **Interest Rate Risk**

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KEYWORDS

- Price effect
- Price Risk
- Interest on interest effect
- Reinvestment Risk

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Market Risk

- **Market Risk** is the uncertainty that the realized return will deviate from the expected return because of interest rate changes.
- The return on a bond comes from: (i) Coupons, (ii) Interest earned from reinvesting coupons: interest on interest and (iii) Capital gains or losses
- A change in rates affects interest on interest and capital gains or losses.

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Market Risk

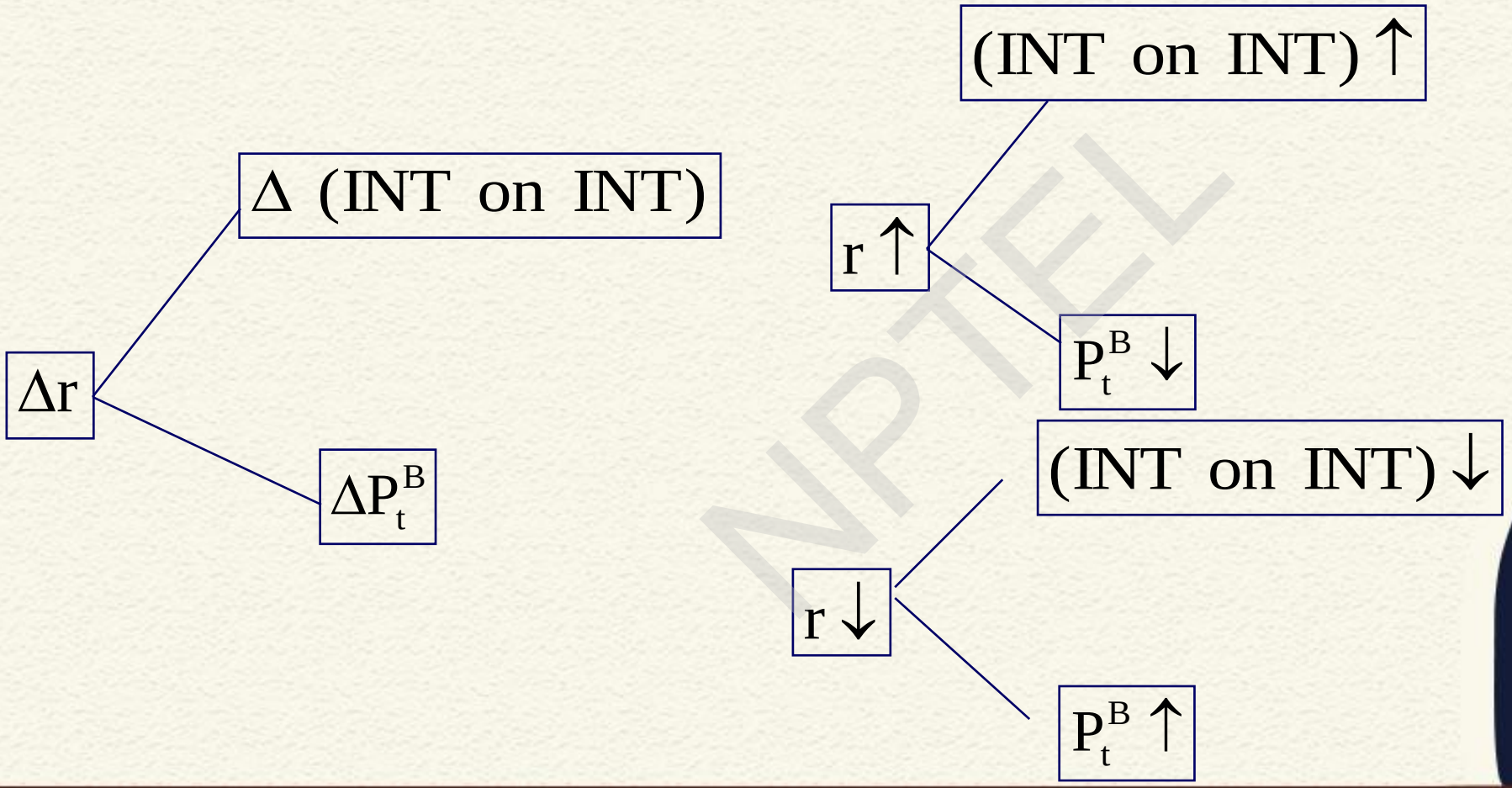
- A change in interest rates has two effects on a bond's return: **price effect** and **interest-on-interest effect**.

- **Price Effect*:*** Interest rate changes affect the price of a bond; this is referred to as **price risk**. If the investor's horizon date, HD, is different from the bond's maturity date, then the investor will be uncertain about the price he will receive from selling the bond (if HD < M), or the price he will have to pay for a new bond (if HD > M).

- **Interest-on-Interest Effect**: Interest rate changes affect the return the investor expects from reinvesting the coupon – **reinvestment risk**. Thus, if an investor buys a coupon bond, he automatically is subject to market risk.

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Market Risk



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Example

- Consider the case of an investor with a horizon of 3.5 years who buys a 10-year, 10% annual coupon bond at its par value of Rs. 1000 to yield 10%.
- If the yield curve were initially flat at 10% and if there were no changes in the yield curve in the ensuing years, then the investor would realize a rate of return of 10%.
- With no change in the flat yield curve, the investor would be able to reinvest each of her coupons at a rate of 10%, yielding a coupon value of Rs. 347.16 at year 3.5.
- The Rs. 347.16 coupon value consists of Rs. 300 in coupons and Rs. 47.16 in interest earned from reinvesting the coupon; that is, **interest on interest** of Rs. 47.16.
- In addition, with no change in the flat yield curve, the investor would be able to sell the original 10-year bond (now with a maturity of 6.5 years) for Rs. 1048.81 at the end of 3.5 years:

$$P_4 = \sum_{t=1}^{6} \frac{100}{(1.10)^t} + \frac{1000}{(1.10)^{10}} = 1000 \qquad P_{3.5} = \frac{1000 + 100}{(1.10)^{0.5}} = 1048.81$$

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Example Cont…

Combined, the selling price of Rs. 1048.81 and the coupon value of Rs. 347.16 yield an HD value of Rs. 1,395.97, which equates to a total return of 10% for the 3.5 years. This is the same rate as the initial YTM:

$$1000 = \frac{1395.97}{(1+TR)^{3.5}}$$

$$TR = \left[\frac{1395.97}{1000}\right]^{1/3.5} - 1 = 0.10$$

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Example Cont…

If there is no change in the yield curve,

then the TR for 3.5 *years is* 10%

$$HDValue = 100(1.10)^{2.5} + 100(1.10)^{1.5} + 100(1.10)^{.5} + P_{3.5}^{B} = 1395.97$$

$$where: P_{3.5}^{B} = \frac{1000 + 100}{(1.10)^{.5}} = 1048.81$$

$$P_{4}^{B} = \sum_{t=1}^{6} \frac{100}{(1.10)^{t}} + \frac{1000}{(1.10)^{6}} = 1000$$

$$TR_{3.5} = \left[\frac{1395.97}{1000}\right]^{1/3.5} - 1 = 0.10$$

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Example Cont…

- Suppose that shortly after the investor purchased the bond, though, the flat 10% yield curve shifted up to 12% and remained there for the 3.5 years.

$$\text{If the yield curve shifts to } 12\%,$$

$$HDValue = 100(1.12)^{2.5} + 100(1.12)^{1.5} + 100(1.12)^{.5} + P_{3.5}^{B} = 1318.81$$

$$where: P_{3.5}^{B} = \frac{917.77 + 100}{(1.12)^{.5}} = 961.70$$

$$P_4^B = \sum_{t=1}^{6} \frac{100}{(1.12)^t} + \frac{1000}{(1.12)^6} = 917.77$$

$$TR_{3.5} = \left[\frac{1318.81}{1000}\right]^{1/3.5} - 1 = 0.0823$$

- At HD the investor would be able to sell the bond for only Rs. 961.70, resulting in a capital loss of Rs. 38.30. This loss would be partly offset, though, by the gains realized from reinvesting the coupons at 12%. Combined, the investor's HD value would be Rs.1318.81, which is Rs. 77.16 less than the HD value of Rs. 1395.97 realized if rates had remained constant at 10%. The TR would be only 8.23%.

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Example Cont…

- If the yield curve had shifted down from 10% to 8% and remained there, then the investor would have gained on the sale of the bond but would have earned less interest from reinvesting the coupons.

If the yield curve shifts to 8%

$$HDValue = 100(1.08)^{2.5} + 100(1.08)^{1.5} + 100(1.08)^{.5} + P_{3.5}^{B} = 1484.82$$

$$where: P_{3.5}^{B} = \frac{1092.46 + 100}{(1.08)^{.5}} = 1147.44$$

$$P_{4}^{B} = \sum_{t=1}^{6} \frac{100}{(1.08)^{t}} + \frac{1000}{(1.08)^{6}} = 1092.46$$

$$TR_{3.5} = \left[\frac{1484.82}{1000}\right]^{1/3.5} - 1 = 0.1196$$

- In this case, the HD value increases to Rs. 1484.82 to yield a TR of 11.96%

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Price Effect and Interest on Interest Effect

- The interest rate change has two opposite effects on total return: (i) Inverse price effect, (ii) Direct interest-on-interest effect

- Whether the total return varies directly or inversely to rate changes depends on which effect dominates.

- In this case (10-year, 10% coupon bond with HD = 3.5 yrs), the inverse price effect dominates. This causes the TR to vary inversely with rate changes.

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Bond with interest-on-interest effect that dominates price effect

If the yield curve shifts to 12%,

$$HD\ Value = 200(1.12)^{2.5} + 200(1.12)^{1.5} + 200(1.12)^{.5} + P_{3.5}^{B} = 1848.12$$

$$where:\ P_{3.5}^{b} = \frac{1000+200}{(1.12)^{.5}} = 1133.89$$

$$TR_{3.5} = \left[\frac{1848.12}{1317}\right]^{1/3.5} - 1 = 0.1016$$

$$where:\ P_{0}^{b} = \sum_{t=1}^{4} \frac{200}{(1.10)^{t}} + \frac{1000}{(1.10)^{4}} = 1317$$

If the yield curve shifts to 8%:

$$HD\ Value = 200(1.08)^{2.5} + 200(1.08)^{1.5} + 200(1.08)^{.5} + P_{3.5}^{B} = 1829.45$$

$$where:\ P_{3.5}^{b} = \frac{1000+200}{(1.08)^{.5}} = 1154.70$$

$$TR_{3.5} = \left[\frac{1829.45}{1317}\right]^{1/3.5} - 1 = 0.09845$$

$$where:\ P_{0}^{b} = \sum_{t=1}^{4} \frac{200}{(1.10)^{t}} + \frac{1000}{(1.10)^{4}} = 1317$$

Suppose the investor purchased a *four-year, 20% annual coupon* bond when the yield curve was flat at 10%

With an HD of 3.5 years, the four-year, 20% bond has an interest-on-interest effect that dominates the price effect, resulting in the direct relationship between the ARR and interest rate changes.

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Bond with interest-on-interest effect that equals price effect

- It is possible to select a bond in which the interest on interest and price effects exactly offset each other.

- When this occurs, the TR will not change when there is a yield curve shift just after the purchase.

Example:

Suppose the investor purchased a *four-year, 9% annual* coupon for Rs. 968.30 to yield 10%.

If the flat yield curve immediately shifted to 12%, 8%, or any other rate, the total return would remain at 10%.

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Bond with interest-on-interest effect that equals price effect

If the yield curve shifts to 12%:

$$HD\ Value = 90(1.12)^{2.5} + 90(1.12)^{1.5} + 90(1.12)^{.5} + P_{3.5}^{B} = 1351.35$$

$$where:\ P_{3.5}^{b} = \frac{1000+90}{(1.12)^{.5}} = 1029.95$$

$$TR_{3.5} = \left[\frac{1351.35}{968.30}\right]^{1/3.5} - 1 = 0.10$$

$$where: P_0^b = \sum_{t=1}^{4} \frac{90}{(1.10)^t} + \frac{1000}{(1.10)^4} = 968.30$$

If the yield curve shifts to 8%:

$$HD\ Value = 90(1.08)^{2.5} + 90(1.08)^{1.5} + 90(1.08)^{.5} + P_{3.5}^{B} = 1352.49$$

$$where:\ P_{3.5}^{b} = \frac{1000+90}{(1.08)^{.5}} = 1048.85$$

$$TR_{3.5} = \left[\frac{1352.49}{968.30}\right]^{1/3.5} - 1 = 0.10$$

$$where:\ P_0^b = \sum_{t=1}^{4} \frac{90}{(1.10)^t} + \frac{1000}{(1.10)^4} = 968.30$$

The investor can do this by buying a bond that pays a coupon and has a maturity different than investor's HD

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CONCLUSIONS

- The interest rate change has two opposite effects on total return: (i) Inverse price effect, (ii) Direct interest-on-interest effect
- Whether the total return varies directly or inversely to interest rate changes depends on which effect dominates.

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REFERENCES

- Johnson, R.S. (2010), Bond Valuation, Selection, and Management, Second Edition, John Wiley & Sons, Inc., Hoboken, New Jersy.
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Lecture 23: Bond Duration

CONCEPTS COVERED

- **Bond Duration**

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KEYWORDS

- Macaulay's duration
- Modified duration
- Portfolio duration

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Duration

- A bond's *duration* (D) can be defined as the weighted average of the bond's time periods, with the weights being each time period's relative present value of its cash flow
- It is the weighted average on a present value basis of the time to full recovery of the principal and interest payments on a bond. It measures the weighted average maturity of a bond’s cash flows on a present value basis.

$$D = \sum_{t=1}^{N} t \frac{PV(CF_t)}{P_0^b}$$

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Duration Cont…

- It is a measure of effective maturity that incorporates the timing and size of a security's cash flows.
- It captures the combined impact of market rate, size of interim payments and maturity on a security's price volatility
- Conceptually, duration is a measure of interest elasticity in determining a security's market value.
- Thus if a security's duration is known, an investor can readily estimate the size of a change in value (or price) for different rate changes

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Duration Cont…

Find out the duration of a 4-year, 9% annual coupon bond with par value Rs. 1000 given a flat yield curve at 10%

t	CF_t	$CF_t/(1.10)^t$	$PV(CF_t)/P^B$	$t[PV(CF_t)/P^B]$
1	90	81.818	.084496	0.084496
2	90	74.380	.076815	0.153630
3	90	67.618	.069832	0.209496
4	1090	744.485	.768857	3.075428
		$P^B = 968.30$		D = 3.52

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Duration of Bond Portfolio

- The duration of a bond portfolio, D_p, is simply the weighted average of each of the bond's durations (D_i), with the weights being the proportion of investment funds allocated to each bond (w_i):

$$D_p = \sum_{i=1} w_i D_i$$

- Instead of selecting a specific bond with a desired duration, an investor could determine the allocations (w_i) for each bond in his portfolio that would yield the desired portfolio duration.

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Duration as a Price Sensitivity Measure

- Though duration is defined as the weighted average of a bond's time periods, it is also an important measure of volatility.
- As a measure of volatility, duration is defined as the percentage change in a bond's price ($\%\Delta P = \Delta P/P_0$) given a small change in yield, dy.
- Mathematically, duration is obtained by taking the derivative of the equation for the price of a bond with respect to the yield, then dividing by the bond's price and expressing the resulting equation in absolute value.

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Duration as a Price Sensitivity Measure Cont…

$$\text{Duration} = \frac{dP/P}{dy} = \frac{1}{(1+y)}\left(\sum_{t=1}^{N} t\frac{PV(CF_t)}{P_0^B}\right)$$

dP/P_0 = percentage change in the bond's price
dy = small change in yield
N = number of periods to maturity (M)

- The bracketed expression is the weighted average of the time periods, defined in the last section as duration.
- Formally, the weighted average of the time periods is called Macaulay's duration, and the equation, which defines the percentage change in the bond's price for a small change in yield in absolute value, is called the modified duration.

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Duration as a Price Sensitivity Measure Cont…

- Thus, the modified duration is equal to Macaulay's duration divided by 1 + y:
- The proportional change in the bond price to a change in yield, (dP/P/dy) is negative (inverse relationship between price and yield).
- The convention, though, is to express Modified duration in absolute value.

$$Modified\,Duration = \frac{1}{(1+y)}\left[Macaulay's\ Duration\right]$$

$$Macaulay's\ Duration = \left(\sum_{t=1}^{N} t\frac{PV(CF_t)}{P_0^B}\right)$$

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Example

The 4-year, 9% annual coupon bond (used in the example) has a Macaulay duration of 3.52 years, and given the initial yield of 10%, a modified duration of 3.20:

$$Modified\,Duration = \frac{1}{(1+y)}\left[Macaulay's\ Duration\right]$$

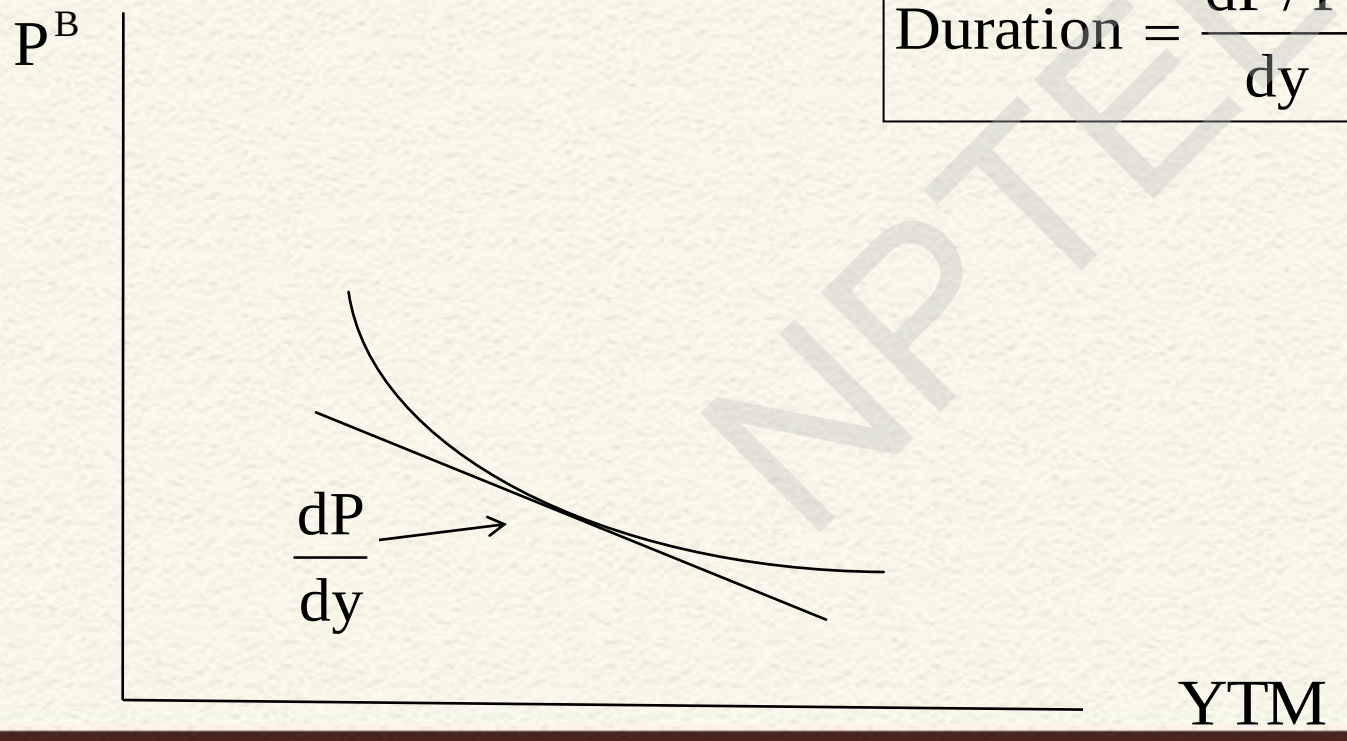
$$Modified\,Duration = \frac{1}{(1.10)}\left[3.52\right] = 3.20$$

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Duration as a Price Sensitivity Measure Cont…

Modified duration is the slope of price-yield curve (dP/dy) divided by P: (dP/dy)(1/P) = dP/P/dy = %ΔP/dy

$$\text{Duration} = \frac{dP/P}{dy} = \frac{\%\Delta P}{dy}$$



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Duration as a Price Sensitivity Measure Cont…

The price of a bond that pays coupons each period and its principal at maturity is

$$P_0^B = C\left[\frac{1-(1/(1+y))^N}{y}\right] + \frac{F}{(1+y)^N}$$

Taking the first derivative of this equation, dividing through by P, and expressing the resulting equation in absolute value provides a measure of duration for a bond paying principal at maturity:

$$\text{Modified Duration} = \frac{\frac{C}{y^2}\left[1 - \frac{1}{(1+y)^N}\right] + \frac{N[F - (C/y)]}{(1+y)^{N+1}}}{P_0^b}$$

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Duration as a Price Sensitivity Measure Cont…

- The measures of duration are defined in terms of the length of the period between payments.
- If the cash flow is distributed annually, duration reflects years; if cash flow is semi-annual, then duration reflects half years.
- The convention is to express duration as an annual measure.
- Annualized duration is obtained by dividing duration by the number of payments per year (n):

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Duration as a Price Sensitivity Measure Cont…

Find the modified duration measured in half-years for a 10-year, 9% coupon bond selling at par (F = 100) with coupon payments made semiannually.

$$\text{Duration in half years} = \frac{\frac{4.5}{0.045^2}\left[1-\frac{1}{(1.045)^{20}}\right]+\frac{20[100-(4.5/0.045)]}{(1.045)^{21}}}{100} = 13$$

$$\text{Annualized Duration} = \frac{13}{2} = 6.5$$

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Properties of Duration

- The lower the coupon rate, the greater the duration.
- The longer the terms to maturity, the greater the duration.
- For zero-coupon bonds, Macaulay's duration is equal to the bond's term to maturity (N) and the modified duration is equal N/(1+y).
- The higher the yield to maturity, the lower the duration. That is, the slope of the price-yield curve is flatter at higher yields.

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Uses of Duration

- Knowing a bond or bond portfolio's duration is important in formulating bond strategies.

- For example, a bond speculator who is anticipating a decrease in interest rates across all maturities (downward parallel shift in the yield curve) could realize a potentially greater expected return, but also greater risk, by purchasing a bond with a relatively large duration.

- In contrast, a bond portfolio manager expecting a parallel upward shift in the yield curve could take defensive actions against possible capital losses by reallocating his portfolio such that it would have a lower portfolio duration.

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Uses of Duration

- Duration is also used as an estimate of the percentage change in a bond's price for a small change in rates

Consider the 10-year, 9% coupon bond selling at 100 to yield 9%. If the yield were to increase by 10 annual basis points (from 9% to 9.10%), then using modified duration equation, the bond would decrease by approximately 0.65%:

$$\%\Delta P^b = \frac{dP^b}{P_0^b} = [Modified\ Duration]dy$$

$$\%\Delta P^b = \frac{dP^b}{P_0^b} = [6.5][0.0910 - 0.09] = 0.0065$$

This is very close to the actual percentage change of −0.6476%.

$$\%\Delta P^b = \frac{99.3524 - 100}{100} = -0.006476$$

$$where: P = \sum_{t=1}^{20} \frac{4.50}{(1.0455)^t} + \frac{100}{(1.0455)^{20}} = 99.3524$$

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Uses of Duration

For example, if the yield had increased by 200 basis points to 11%, instead of only 10 basis points, the approximate percentage change using the duration measure would be −13%:

$$\%\Delta P^b = [6.5][0.11 - 0.09] = 0.13$$

This contrasts with the actual percentage change of −11.95%:

$$\%\Delta P^b = \frac{88.0496 - 100}{100} = -0.1195$$

$$where: P = \sum_{t=1}^{20} \frac{4.50}{(1.055)^t} + \frac{100}{(1.055)^{20}} = 88.0496$$

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Uses of Duration

- The greater the change in yields, the less accurate duration is in estimating the approximate percentage change in price
- In addition, using duration to estimate discrete changes does not capture the asymmetrical gain and loss relation that characterizes the price-yield curve.
- For yield increases, duration overestimates the price change
- For yield decreases, duration underestimates the price change

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Portfolio Duration

Bond	Annual Coupon Rate	Maturity	Price	Market Value (Rs)	Portfolio Weights	Modified Duration	Weighted Duration
A	0	3 yrs	80.18	2,00,00,000	0.0833	2.8916	0.2410
B	10.00%	3 yrs	106.61	3,00,00,000	0.1250	2.5800	0.3225
C	10.00%	5 yrs	110.27	4,00,00,000	0.1667	3.9571	0.6595
D	5.00%	10 yrs	82.63	6,00,00,000	0.2500	7.4675	1.8669
E	10.00%	10 yrs	117.37	9,00,00,000	0.3750	6.5833	2.4687
				24,00,00,000	1.0000		5.5586

- The portfolio has a modified duration of 5.5586. If the yields on each of the bonds change by 100 bp, the portfolio value would change by approximately 5.5586%.
- Each bond's weighted duration, $w_i D_i$, measures that bond's contribution to the overall portfolio's duration.
- Bond A contributes 0.2410/ 5.5586 = 4.335% to the portfolio's duration of 5.5586 and Bond D = contributes = (1.8669/5.5586) 33.58% .

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CONCLUSIONS

- Duration of a bond measures the weighted average maturity of a bond’s cash flows on a present value basis
- duration is defined as the percentage change in a bond's price given a small change in yield
- The lower the coupon rate, the greater the duration and the longer the terms to maturity, the greater the duration.
- For zero-coupon bonds, Macaulay’s duration is equal to the bond’s term to maturity
- The modified duration is equal to Macaulay's duration divided by (1 + yield)

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- Johnson, R.S. (2010), Bond Valuation, Selection, and Management, Second Edition, John Wiley & Sons, Inc., Hoboken, New Jersy.
- Fabozzi, J. Frank and Mann, V. Steven (2005): The Hand Book of Fixed Income Securities, Tata McGraw-Hill, 7th Edition.

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Management of Fixed Income Securities

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Lecture 24: Bond Convexity

CONCEPTS COVERED

- **Bond Convexity**

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KEYWORDS

- Annualized convexity
- Convexity bias
- Approximate duration
- Approximate convexity

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Convexity

- The price-yield curve is not linear, but convex from below (bowed-shaped).
- Duration is a measure of the slope of the price-yield curve at a given point – *first-order derivative*.
- ***Convexity*** is a measure of the change in the slope of the price-yield curve – *second-order derivative*.
- Convexity measures how bowed-shaped the price-yield curve is

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Convexity Cont…

- Convexity means that the slope of the price-yield curve (Δ = dP/dy) gets smaller as you move down the curve or as the YTM increases

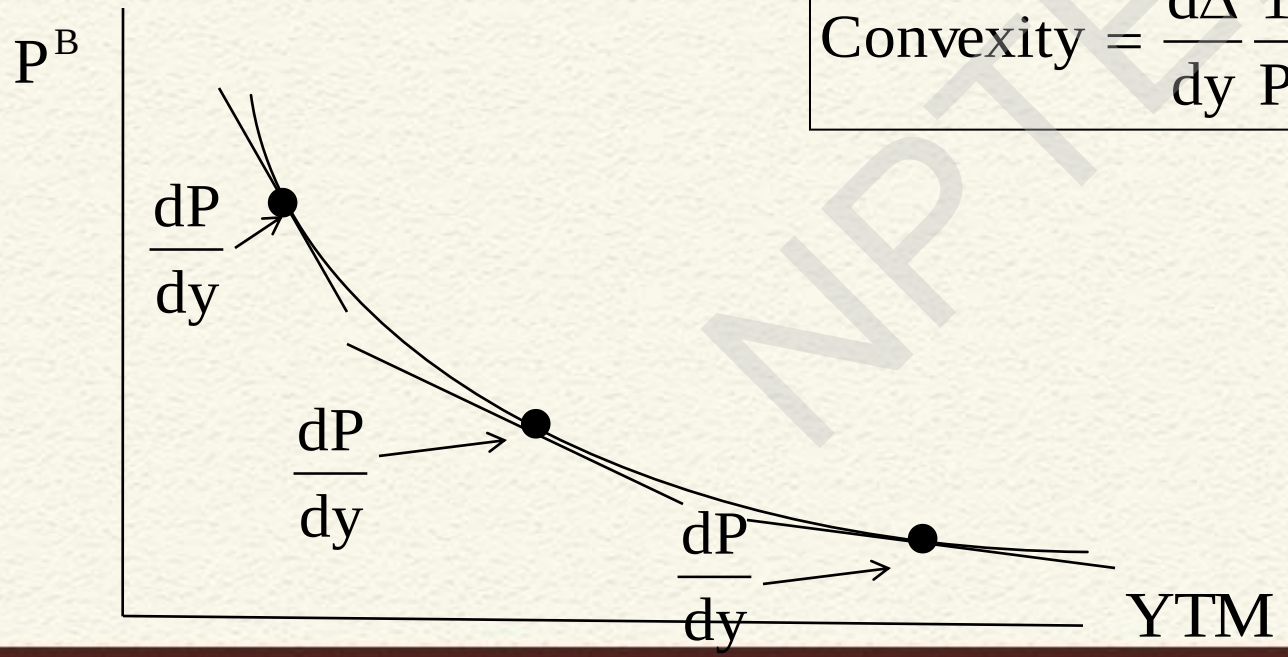
- This implies that for a given absolute change in yields, the percentage increase in price will be greater in absolute value for the yield increase than the percentage decrease in price in absolute value for the yield decrease.

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Convexity Cont…

- Convexity is the change in slope (Δ) of Price-Yield curve (d(dP/dy)/dy = dΔ/dy) divided by P

$$\text{Convexity} = \frac{d\Delta}{dy}\frac{1}{P}$$



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Convexity Cont…

- For an investor who is long in a bond, its convexity suggests that the capital gain resulting from a decrease in rates will be greater than the capital loss resulting from an increase in rates of the same absolute magnitude.
- Bonds or bond portfolio with greater convexity have a greater asymmetrical gain-loss relation.
- All other things equal, the greater a bond convexity the more valuable the bond.

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Measurement of Convexity

Mathematically, convexity is the change in the slope of the price-yield curve for a small change in yield; it is the second-order derivative.

It is derived by taking the derivative of equation for dP/dy and dividing the resulting equation by the current price.

Doing this yields:

$$\text{Convexity} = \frac{1}{P_0^b}\left[\sum_{t=1}^{N}\frac{t(t+1)(CF_t)}{(1+y)^{t+2}}\right]$$

The convexity of a bond that pays a fixed coupon each period and the principal at maturity is

$$\text{Convexity} = \frac{\frac{2C}{y^3}\left[1-\frac{1}{(1+y)^N}\right]-\frac{2CN}{y^2(1+y)^{N+1}}+\frac{N(N+1)[F-(C/y)]}{(1+y)^{N+2}}}{P_0^b}$$

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Measurement of Convexity Cont…

- Like duration, convexity reflects the length of periods between cash flows.
- The annualized convexity is found by dividing convexity, measured in terms of n-payments per year, by n^2:

$$\text{Annualized Convexity} = \frac{\text{Convexity for bond with n – payments per year}}{n^2}$$

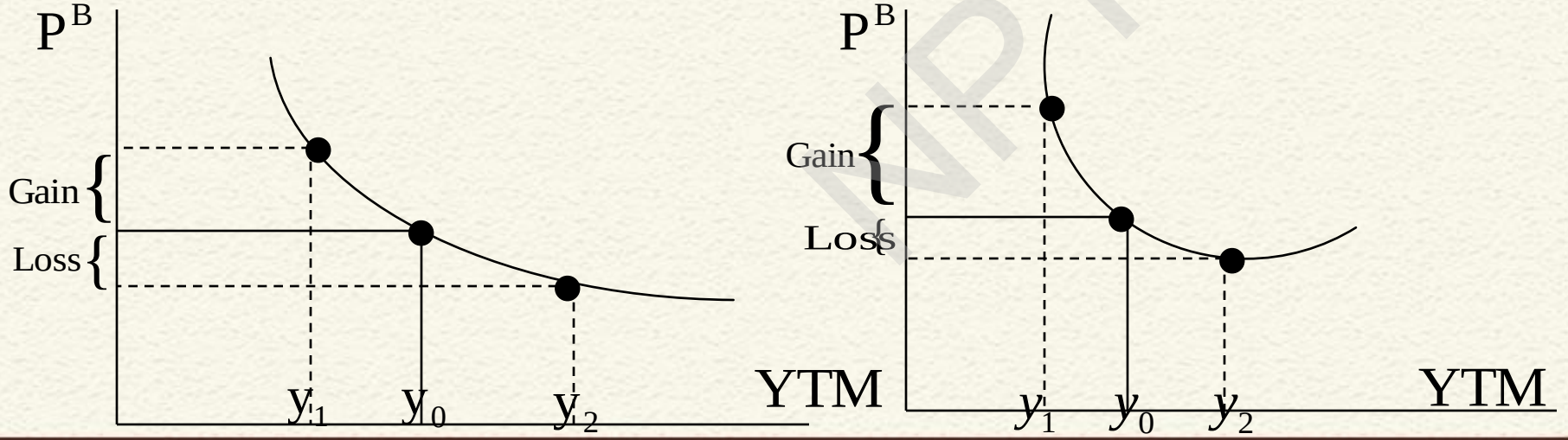
What is the convexity for the 10-year, 9% coupon bond with semi-annual payments?

$$\textit{Convexity}\ (1/2\ \textit{yrs}) = \frac{\frac{2(4.5)}{0.045^3}\left[1-\frac{1}{(1.045)^{20}}\right]-\frac{2(4.5)(20)}{(0.045)^2(1.045)^{21}}+\frac{(20)(21)[100-(4.5/.045)]}{(1.045)^{22}}}{100} = 225.43$$

$$\textit{Annualized Convexity} = \frac{225.43}{2^2} = 56.36$$

Properties and Uses of Convexity

- ***Asymmetrical Gain/Loss Relation***: The greater a bond's convexity, the greater its capital gains and the smaller its capital losses for given absolute changes in yields



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Properties and Uses of Convexity

- Convexity also can be used with duration to estimate the percentage change in a bond's price given a change in yield.

- Unlike duration, which can only provide a good estimate when the yield changes are small, incorporating convexity allows for better estimate of large yield changes.

- The formula for estimating the percentage change in price for a large change in yields is derived using Taylor expansion. This expansion yields:

$$\%\Delta P^{b} = [\text{Modified Duration}]\Delta y + \frac{1}{2}[\text{Convexity}](\Delta y)^{2}$$

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Properties and Uses of Convexity

- The estimated percentage change in the price of our 10-year, 9% coupon bond given a 200 basis point increase in yields is 11.87%

$$\%\Delta P^b = [-6.5](0.02) + \frac{1}{2}[56.36](0.02)^2 = -0.1187$$

- The 11.87% decrease is closer to the actual decrease of 11.95% than the estimated 13% decrease obtained using the duration measure.

- The above formula also results in non-symmetrical percentage increases and decreases.

- For example, if rates had decreased by 200 basis points, the percentage increase would be 14.13%, not 13% that the duration measure yields:

$$\%\Delta P^b = [-6.5](-0.02) + \frac{1}{2}[56.36](0.02)^2 = 0.1413$$

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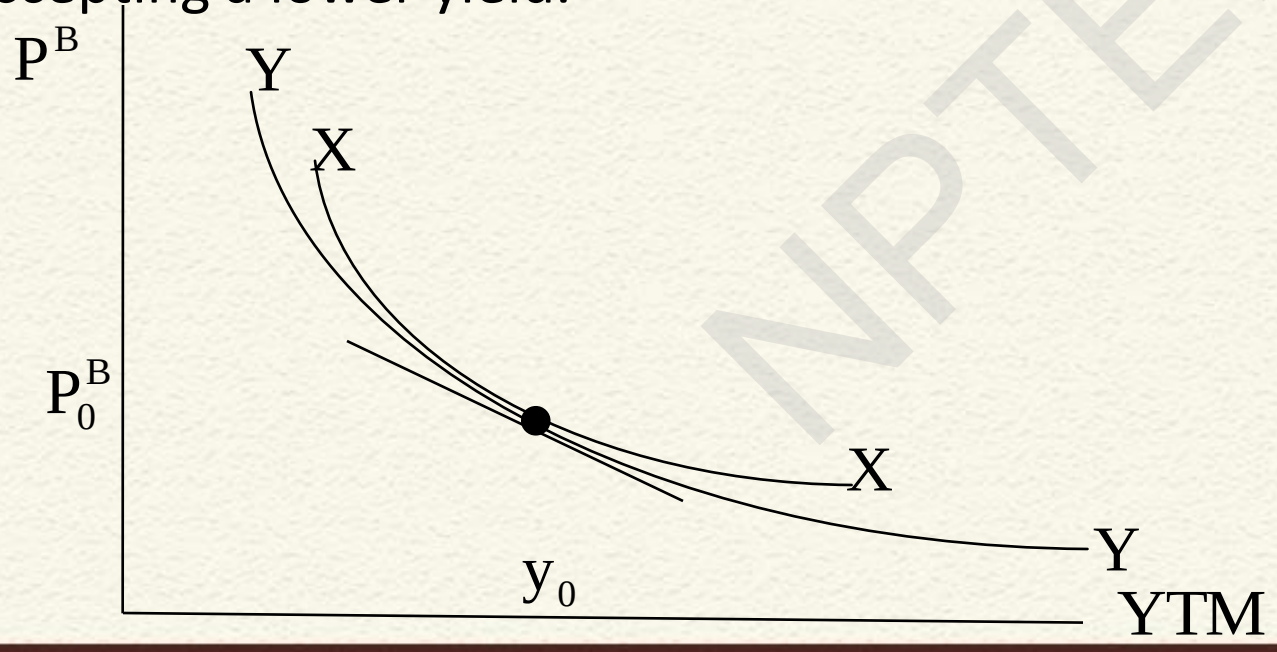
Properties and Uses of Convexity

- As the yield increases (decreases), the convexity of the bond decreases (increases). This is referred to as positive convexity.
- For a given yield and maturity, the lower the coupon, the greater the convexity.
- For a given yield and modified duration, the lower the coupon, the smaller the convexity.

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Value of Convexity

- Consider two similar bonds – X and Y – that are trading at the same yield (y_0) and have the same duration, but with Bond X having a greater convexity than Bond Y.
- In an efficient market, investors would take the greater convexity of X into account, pricing it higher than Y and accepting a lower yield.



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Value of Convexity

- The question is how much would the market pay for the convexity?
- The answer depends on how much the market expects rates to change.
- If investors expect yields to change very little – market with low interest rate volatility – then the advantage of X over Y is small. Thus, in this market, investor would pay little for convexity.
- On the other hand, if investors expect yields to change significantly –a market with high interest rate volatility – then the advantage of X over Y is more important. Thus, in this market, investor would pay more for convexity.
- Given that longer term bonds have greater duration and therefore volatility, a priori, one would expect the yields on longer term bond to be affected more by convexity than shorter term bonds.
- The influence of convexity on the shape of the yield curve and the relation between returns and duration is referred to as the ***convexity bias***.

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Alternative Duration and Convexity Formulas

- Duration and convexity can also be estimated by determining the price of the bond when the yield increases by a small number of basis points (e.g., 2-10 basis points), P_+, and when the yield decreases by the same number of basis points, P_-

- These measures are referred to as ***approximate duration*** and ***convexity*** and can be estimated using the following formulas:

$$\text{Approx. Duration} = \frac{P_- - P_+}{2(P_0)(\Delta y)}$$

$$\text{Approx. Convexity} = \frac{P_+ + P_- - 2P_0}{P_0(\Delta y)^2}$$

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CONCLUSIONS

- **Convexity** is a measure of the change in the slope of the price-yield curve
- The greater a bond's convexity, the greater its capital gains and the smaller its capital losses for given absolute changes in yields
- The influence of convexity on the shape of the yield curve and the relation between returns and duration is referred to as the ***convexity bias***
- Given that longer term bonds have greater duration and one would expect the yields on longer term bond to be affected more by convexity than shorter term bonds.

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REFERENCES

- Johnson, R.S. (2010), Bond Valuation, Selection, and Management, Second Edition, John Wiley & Sons, Inc., Hoboken, New Jersy.
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Management of Fixed Income Securities

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Lecture 25: Default Risk

CONCEPTS COVERED

- **Default Risk or Credit Risk**
- **Credit Spread**

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KEYWORDS

- Credit Rating
- Default loss rate
- Recovery rate
- Credit watch
- Ratings transition matrix

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Default Risk

- Default Risk or credit risk is the uncertainty that the realized return may deviate from the expected return due to the failure of the issuer to meet the contractual obligations specified in the indenture.
- A failure to meet any of the interest payments, the principal obligation, or other terms specified in the indenture places the borrower/issuer in default.

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Default Ratings

- Many large institutional investors have their own credit analysis departments to evaluate bond issues in order to determine the abilities of companies to meet their contractual obligations.
- Institutional and individual bond investors do rely on bond rating companies.

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Credit Rating Agencies in India and Abroad

India

- Credit Rating and Information Services of India Ltd. (**CRISIL)**
- Investment information and Credit Rating Agency Ltd. **(ICRA)**
- Credit Analysis and Research Ltd. **(CARE)**
- Brickwork Ratings India Private Limited (BWR)
- SME Rating Agency of India Ltd. (SMERA**)**

Global Players

- Moody's
- Standard & Poor
- Fitch Ratings

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CRISIL Debenture Rating Symbols

Symbol	Safety Level
CRISIL AAA	Highest Safety
CRISIL AA	High Safety
CRISIL A	Adequate Safety
CRISIL BBB	Moderate Safety
CRISIL BB	Inadequate Safety
CRISIL B	High Risk
CRISIL C	Substantial Risk
CRISIL D	Default

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Moody's and Standard & Poor's Rating Symbols

Moody's (Symbols)	Standard & Poor (Symbols)	Safety Level
Aaa, Aa	AAA, AA	Very high quality
A, Baa	A, BBB	High Quality
Ba, B, Caa, Ca	BB, B, CCC, CC	Speculative
C, D	C, D	Very poor

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Credit Rating Methodology

- Business Analysis
- Financial Analysis
- Management Evaluation
- Geographical Analysis
- Regulatory & Competitive Environment
- Fundamental Analysis

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Types of Bonds on the Basis of Ratings

- Bonds with relatively low chance of default are referred to as investment-grade bonds, with quality ratings of Baa (or BBB) or higher.

- Bonds with relatively high chance of default are referred to as non-investment grade, speculative-grade or junk bonds and have quality ratings below Baa.

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Probability Intensities

- **Cumulative Default Rates**: the default chance through time.
- **Unconditional Probability Rates**: These are the probabilities of default in a given year as viewed from time zero. The unconditional probability of a bond defaulting during year t is equal to the difference in the cumulative probability in year t minus the cumulative probability of default in year t −1.
- **Conditional Probability Rates**: It is the probability of default in a given year conditional on no prior defaults. Conditional probabilities of default are known as ***probability intensities***. This probability is equal to unconditional probability of default in time t as a proportion of the bond's probability of survival at the beginning of the period. The probability of survival is equal to 100 minus the cumulative probability

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Probability Intensities (Example)

Bond Rating: A		Year				
		1	2	3	4	5
	Cumulative Probability	0.05	0.16	0.22	0.34	0.45
	Unconditional Probability		0.11	0.06	0.12	0.11
	Conditional Probability		0.130952	0.076923	0.181818	0.20000

Cumulative Probability:

For example, the A corporate bonds have:

5% chance of defaulting after one year
16% chance after two years
22% chance after three years
34% chance after 4 years.

Unconditional Probability: As shown in the table, the probability of a A bond default during year 3 is equal to 6% (= 22% − 16%).

Conditional Probability: The probability that a A bond will survive until the end of year three is 78% (100 minus its cumulative probability 22%), and the probability that the A bond will default during year 4 conditional on no prior defaults is 13.09% (= 6%/78%).

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Default Loss Rate

- Holders of defaulted bonds usually recover a percentage of their investment: recovery rate
- Default Loss Rate = Default Rate(1 – Recovery Rate)
- For example, if the recovery rate from a defaulted bond is estimated to be 40% and the bond's default rate is 8%, then the default loss rate would be 4.2%:
- Default Loss Rate = 8% (1 − 0.40) = 4.8%

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Downgrades and Ratings Transition Matrix

- Credit risk involves not only the uncertainty related to actual default, but also the concern that the bond's default risk may increase leading to a downgrade in its quality rating and decrease in the bond's price.
- Usually, when an issuer is placed on a ***credit watch***, it is for a downgrade not an upgrade
- Rating agencies also accumulate statistics on how ratings change over time.
- The information is presented in a ***ratings transition matrix***.
- The matrix is defined in terms of quality ratings at the beginning of the year (column) and ratings at the end of the year (row).

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Downgrades and Ratings Transition Matrix (Example)

	Ratings at the End of the Year							
Ratings at the Start of the Year	AAA	AA	A	BBB	BB	B	CCC -D	Total
AAA	92%	7%	0.5%	0.5%	0	0	0	100%
AA	1%	94%	4%	0.5%	0.5%	0	0	100%
A	2%	2%	89%	5%	1%	1%	0	100%
BBB	0	1%	2.5%	87%	6%	3%	0.5%	100%
BB	0	0	2%	3%	86%	6%	3%	100%

94% of the AA bonds at the beginning of the year remained that way at the end of the year

1% of AA bonds at the beginning of the year were upgraded to AAA by the end of the year

4% of AA bonds at the beginning of the year were downgraded to A by the end of the year.

Credit Spread

- A credit spread on a bond can be thought of as the bond investor's expected loss from the principal from default.
- Consider a portfolio of 7-year BBB bonds trading at a 3% credit spread.
- The 2% premium that investors receive from the bond portfolio represents their compensation for an implied expected loss of 2% per year of the principal from the defaulted bonds.
    - If the spread were 3% and bond investors believed that the expected loss from default on such bonds would be only 2% per year of the principal, then the bond investors would want more BBB bonds, driving the price up and the yield down until the premium reflected a 1% spread.
    - If the spread were 3% and bond investors believed the default loss on a portfolio of BBB bonds would be 4% per year, then the demand and price for such bonds would decrease, increasing the yield to reflect a credit spread of 4%.

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Credit Spread

- Over a period of time, probabilities of default and associated spreads will change (increasing or decreasing) depending on the quality of the credit.
- Usually adverse economic conditions result in greater default probabilities and credit spreads.
  - For corporate bonds, such developments could be aggregate economic factors such as a recession; industry factors like declining sales due to competition; or firm specific factors related to investment or financing decisions.
  - For other bonds like municipal bonds, adverse economic developments include declining property values, municipal government deficits, increasing regional unemployment, or increased use of debt reserves.

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Event Risk

- The default risk of an issuer can change unexpectedly because of any event:
  - Natural or industrial accident
  - Regulatory change
  - Takeover
  - Corporate restructuring

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CONCLUSIONS

- Credit risk is the uncertainty that the realized return may deviate from the expected return due to the failure of the issuer to meet the contractual obligations specified in the indenture
- Bond investors rely rating agencies while investing in the bonds
- A credit spread on a bond can be thought of as the bond investor's expected loss from the principal from default
- The default risk may change due to some external events

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- Johnson, R.S. (2010), Bond Valuation, Selection, and Management, Second Edition, John Wiley & Sons, Inc., Hoboken, New Jersy.
- Fabozzi, J. Frank and Mann, V. Steven (2005): The Hand Book of Fixed Income Securities, Tata McGraw-Hill, 7th Edition.

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